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ANALYSIS OF WAVES IN BOUNDED MAGNETOACTIVE PLASMA

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The investigation dealt with here is an analysis of radially propagating waves in a bounded magnetoactive plasma. The configuration conceived for this analytical investigation is a partially ionized plasma slab, bounded by two infinitely large conducting plates of finite separation. The plates may be in direct contact with the plasma or insulated from it by thin layers of dielectric materials. The static magnetic field is imposed upon the plasma in a direction normal to the plates. The waves are generated by a current source in the plasma with an arbitrary current distribution. This portion of the report describes the preliminary study that has been carried out.

Although the main theme of the proposed analytical problem is quite specific, in the course of the analysis an effort has been made to utilize techniques as general as possible in hope that the methods may be employed elsewhere to solve other types of problems as well.

I. Homogeneous Wave Equation in Anisotropic Media

The presence of a static magnetic field in a plasma region results in an effective electrical conductivity of the plasma which is a dyadic form. Assuming monochromatic waves, the wave equation may be written as:

$$\nabla \times \nabla \times \vec{E} - \vec{\vec{k}} \cdot \vec{E} = 0 \qquad (2\text{-}1)$$

where $\vec{E}$ is the electric field intensity. Assuming spatial homogeneity, the dyadic $\vec{\vec{k}}$ is a function of the frequency, the permittivity, the permeability, the magnetic field, and the effective electrical conductivity of the plasma only.

Equation (2-1) is seen to resemble a Helmholtz equation except that $\vec{\vec{k}}$ is a dyadic; it may be called a dyadic-vector Helmholtz equation. It is well known that the scalar Helmholtz equation is separable into eleven different coordinate systems, and that the vector Helmholtz equation is separable into only six coordinate systems.[1] Despite the fact that the dyadic-vector Helmholtz equation has been frequently encountered in connection with the studies of crystal materials and plasma fields, and that its solutions have been obtained and used extensively for problems involving the rectangular coordinate system and the circular cylindrical coordinate system,[2] to this author's knowledge the separability of the dyadic-vector Helmholtz equation has not yet been fully investigated. In this work, the separability of the dyadic-vector Helmholtz equation is studied, since by determining the coordinate systems in which the equation is separable one has gained knowledge of exactly in what coordinate systems the equation is solvable by a separation method. In the application of boundary value problems, separation into the form that facilitates the fitting of boundary surfaces is most desirable. Hence, it is advisable to separate the dyadic-vector Helmholtz equation into terms of longitudinal, $\vec{L}$, and transverse, $\vec{M}$ and $\vec{N}$, vector components.

The first term in Eq. (2-1) is a vector operator term. A review of the separability of a vector Helmholtz equation shows that the coordinate systems in which the vector Helmholtz equation is separable must be a coordinate system in which one of the scale factors is unity, and that the ratio of the other two scale factors must be independent of the coordinate corresponding to the unity scale factor. The six coordinate systems which meet these requirements are the spherical, the conical, and the four cylindrical coordinate systems. If the first term of Eq. (2-1) is to follow a similar pattern of separation, the coordinate systems in which Eq. (2-1) is separable must fulfill the same requirement that the separable coordinate systems for vector Helmholtz equation fulfill.

The dyadic $\overrightarrow{\overrightarrow{k}}$ in Eq. (2-1) is by no means an arbitrary constant. For the purpose of this work, $\overrightarrow{\overrightarrow{k}}$ may be limited to the form:

$$\overrightarrow{\overrightarrow{k}} = \begin{pmatrix} k_\perp & k_T & 0 \\ -k_T & k_\perp & 0 \\ 0 & 0 & k_{\parallel} \end{pmatrix} \qquad (2\text{-}2)$$

Eq. (2-2) implies that the static magnetic field is in the direction parallel or anti-parallel to the coordinate corresponding to the unity scale factor. Without losing generalities, one denotes this coordinate ξ_3 , and its unit vector $\vec{a}_3$. A close examination shows that only four out of the six coordinate systems are physically realizable for such alignment of the static magnetic field; namely, the four cylindrical coordinate systems including the rectangular, the circular cylindrical, the elliptical cylindrical, and the parabolic cylindrical coordinate systems. In either system, ξ_3 corresponds to the z axis. It may first seem to be pessimistic that the number of permissible coordinate systems has been reduced to only four right at the onset. Fortunately, however, it turns out that no other restrictions will be imposed that will further reduce the number of permissible coordinate systems.

In attempting to find the solution of Eq. (2-1), the difficulty lies in the fact that the equation is a purely transverse one, while due to the dyadic $\vec{\vec{k}}$, the vector field $\vec{E}$, in general, is not entirely transverse. One may assume that:

$$\vec{E} = \vec{A} - \nabla\phi \tag{2-3}$$

and

$$\nabla\cdot\vec{A} = 0 \tag{2-4}$$

Expanding $\vec{\vec{k}}\cdot\vec{E}$ into vector forms, Eq. (2-1) becomes:

$$\nabla_\perp^2\vec{A}_\perp + (\nabla_\perp^2 A_\parallel)\vec{a}_3 + \frac{\partial^2 A_\parallel}{\partial\xi_3^2}\vec{a}_3 + k_\perp\vec{A}_\perp + k_T\vec{A}_\perp\times\vec{a}_3 + k_\parallel A_\parallel\vec{a}_3$$
$$= k_\perp\nabla_\perp\phi + k_T\nabla_\perp\phi\times\vec{a}_3 + k_\parallel\frac{\partial\phi}{\partial\xi_3}\vec{a}_3 \tag{2-5}$$

It is also recognized that Eq. (2-1) implies:

$$k_T \vec{a}_3 \cdot (\nabla_\perp \times \vec{A}_\perp) + (k_\parallel - k_\perp)\frac{\partial A_\parallel}{\partial \xi_3} = k_\perp \nabla_\perp^2 \phi + k_\parallel \frac{\partial^2 \phi}{\partial \xi_3^2} \tag{2-6}$$

Equations (2-5) and (2-6) constitute a set of basic equations with ξ_3 being the explicit coordinate. The subscript $\perp$ indicates the components of operators or vectors which are perpendicular to $\vec{a}_3$, whereas $\parallel$ indicates those parellel to $\vec{a}_3$.

It is advisable to further separate the transverse vector $\vec{A}$ into two components, one tangential to the $\xi_3 = C$ surface and the other with a component normal to the surface. Thus:

$$\left.\begin{aligned} \vec{A} &= \vec{M} + \vec{N} \\ \vec{M} &= \nabla_\perp \psi \times \vec{a}_3 \\ \vec{N} &= \nabla_\perp \frac{\partial \chi}{\partial \xi_3} - \nabla_\perp^2 \chi \vec{a}_3 \end{aligned}\right\} \tag{2-7}$$

Substitute Eq. (2-7) into Equations (2-5) and (2-6), after some manipulations obtain the following set of equations:

$$\nabla_\perp^2(\nabla_\perp^2 \psi) + \frac{\partial^2}{\partial \xi_3^2}(\nabla_\perp^2 \psi) + k_T \frac{\partial}{\partial \xi_3}(\nabla_\perp^2 \chi) - k_T \nabla_\perp^2 \phi = 0 \tag{2-8}$$

$$\nabla_\perp^2(\nabla_\perp^2 \chi) + \frac{\partial^2}{\partial \xi_3^2}(\nabla_\perp^2 \chi) + k_\parallel(\nabla_\perp^2 \chi) + k_\parallel \frac{\partial \phi}{\partial \xi_3} = 0 \tag{2-9}$$

$$k_T(\nabla_\perp^2 \psi) + (k_\parallel - k_\perp)\frac{\partial}{\partial \xi_3}(\nabla_\perp^2 \chi) + k_\perp \nabla_\perp^2 \phi + k_\parallel \frac{\partial^2 \phi}{\partial \xi_3^2} = 0 \tag{2-10}$$

where Equations (2-8) and (2-9) are the $\perp$ component and $\parallel$ component of Eq. (2-5), respectively.

Close examination of Equations (2-8) to (2-10) shows that solutions may be obtained if the three scalar functions each satisfies:

$$\left.\begin{aligned} \nabla^2\psi + T^2\psi &= 0 \\ \nabla^2\chi + T^2\chi &= 0 \\ \nabla^2\phi + T^2\phi &= 0 \end{aligned}\right\} \qquad (2\text{-}11)$$

or

$$\left.\begin{aligned} \nabla_\perp^2\psi + (T^2 - k_m^2)\psi &= 0 \\ \nabla_\perp^2\chi + (T^2 - k_m^2)\chi &= 0 \\ \nabla_\perp\phi + (T^2 - k_m^2)\phi &= 0 \end{aligned}\right\} \qquad (2\text{-}12)$$

and

$$\left.\begin{aligned} \frac{\partial^2\psi}{\partial\xi_3^2} + k_m^2\psi &= 0 \\ \frac{\partial^2\chi}{\partial\xi_3^2} + k_m^2\chi &= 0 \\ \frac{\partial^2\phi}{\partial\xi_3^2} + k_m^2\phi &= 0 \end{aligned}\right\} \qquad (2\text{-}13)$$

where k_m^2 is the separation constant for the separation of ξ_3 . The argument T^2 in Eq. (2-11) or Eq. (2-12) must satisfy a determinant equation:

$$\begin{vmatrix} (T^2 - k_\perp) & -k_T & k_T \\ 0 & (T^2 - k_m^2)(T^2 - k_\parallel) & -k_\parallel k_m^2 \\ k_T(T^2 - k_m^2) & (k_\parallel - k_\perp)(T^2 - k_m^2) & k_\perp(T^2 - k_m^2) + k_\parallel k_m^2 \end{vmatrix} = 0 \qquad (2\text{-}14)$$

The order of Eq. (2-14) appears to be too high to be readily solved at first, but it turns out that if $\overrightarrow{\overrightarrow{k}}$ is not a function of T^2 the resulting secular equation is only of fourth order, since the other four roots, $T^2 = 0$ and $T^2 = k_m^2$, are trivial and may be discarded. The dispersion relation yielded by Eq. (2-14) is:

$$(T^2 - k_\perp)(T^2 - k_m^2)\left\{k_T^2 - k_\perp(T^2 - k_\perp)\right\} - k_\parallel k_m^2 \left\{k_T^2 + (T^2 - k_\perp)^2\right\} = 0 \qquad (2\text{-}15)$$

Equations (2-8) to (2-10) also indicate that the three scalar functions, ψ, χ, and ϕ, are not independent functions. If the solutions are written as:

$$\left.\begin{aligned} \psi &= A\, f_\psi(\xi_1)\, g_\psi(\xi_2)\, h_\psi(\xi_3) \\ \chi &= B\, f_\chi(\xi_1)\, g_\chi(\xi_2)\, h_\chi(\xi_3) \\ \phi &= C\, f_\phi(\xi_1)\, g_\phi(\xi_2)\, h_\phi(\xi_3) \end{aligned}\right\} \qquad (2\text{-}16)$$

then it is always possible to derive from Equations (2-8) to (2-10) two algebraic equations relating the three arbitrary constants, A, B, and C.

In the case of circular coordinates with open boundary in r and θ, the solutions may be written for an outgoing wave:

$$\left.\begin{aligned} \psi_{mn} &= A_{mn}\, H_n^{(2)}(\sqrt{T^2 - k_m^2}\, r)\, e^{jn\theta} \sin k_m z \\ \chi_{mn} &= B_{mn}\, H_n^{(2)}(\sqrt{T^2 - k_m^2}\, r)\, e^{jn\theta} \cos k_m z \\ \phi_{mn} &= C_{mn}\, H_n^{(2)}(\sqrt{T^2 - k_m^2}\, r)\, e^{jn\theta} \sin k_m z \end{aligned}\right\} \qquad (2\text{-}17)$$

and the functional relations are:

$$B_{mn}\left[(k_\perp - k_m^2)A_{mn} - k_T(k_m B_{mn} + C_{mn})\right]\left[k_T(k_m B_{mn} + C_{mn}) - (k_\perp - k_\parallel)A_{mn}\right]$$

$$= k_\parallel k_m A_{mn} C_{mn} \tag{2-18}$$

and

$$(k_T k_\perp - k_m^2)A_{mn}^2 + \left\{(k_\perp + k_\parallel)k_m^2 A_{mn} + k_\perp^2 A_{mn} - k_\perp k_T(k_m B_{mn} + C_{mn})\right\}C_{mn}$$

$$= (k_m B_{mn} + C_{mn})k_T^2 A_{mn} + B_{mn}\left\{(k_\perp - k_m^2)A_{mn} - k_T(k_m B_{mn} + C_{mn})\right\} \tag{2-19}$$

$$\times k_m(k_\parallel - k_\perp)$$

The eigenvalues n are integers whereas k_m is:

$$k_m = \frac{m\pi}{2a}, \qquad m = 0, 1, 2 \cdots$$

where a is the separation distance between the plates.

III. Inhomogeneous Wave Equation and the Green's Function

When a source is present in the plasma region, the source is represented by a source function $\vec{J}_s$. Eq. (2-1) should be written:

$$\nabla \times \nabla \times \vec{E} - \vec{\vec{k}} \cdot \vec{E} = \vec{J}_s \tag{3-1}$$

It can be shown that Eq. (3-1) is solvable in terms of an integral representation

$$\vec{E}(\vec{r}) = \int_{v_0} \overrightarrow{\overrightarrow{G}}(\vec{r}/\vec{r}_0) \cdot \vec{J}_s(\vec{r}_0)\, dv_0 \qquad (3\text{-}2)$$

where the kernel $\overrightarrow{\overrightarrow{G}}(\vec{r}/\vec{r}_0)$ is the usual Green's dyadic function except that instead of satisfying Eq. (3-1) with an impulse source, it satisfies the following:

$$\nabla \times \nabla \times \overrightarrow{\overrightarrow{G}} - \overrightarrow{\overrightarrow{k}}^{*} \cdot \overrightarrow{\overrightarrow{G}} = \overrightarrow{\overrightarrow{I}}\, \delta(\vec{r} - \vec{r}_0) \qquad (3\text{-}3)$$

where $\overrightarrow{\overrightarrow{I}}$ is the idemfactor and $\overrightarrow{\overrightarrow{k}}^{*}$ is the conjugate of $\overrightarrow{\overrightarrow{k}}$. The use of the conjugate of $\overrightarrow{\overrightarrow{k}}$ in Eq. (3-3) is necessary if it is wished to include the cases where $\overrightarrow{\overrightarrow{k}}$ is not Hermitian. Of course, when $\overrightarrow{\overrightarrow{k}}$ is the Hermitian, $\overrightarrow{\overrightarrow{k}}^{*} = \overrightarrow{\overrightarrow{k}}$. In addition to satisfying Eq. (3-3), the Green's function must also satisfy the same boundary condition that the field satisfies.

One technique using tensor relations leading to an integral representation of the Green's dyadic was first developed by Bunkin[3] and was extended by Chow.[4,5] However, an integral representation of the Green's dyadic has two distinct disadvantages: (1) it is not always easy to evaluate an integral, (2) the integral representation of the Green's function is applicable to problems of infinite domain only. When boundaries exist, it is desirable to construct the Green's function from series of eigenfunctions, which describe the free waves.

The derivation of a Green's function to be discussed here depends upon whether there are boundary surfaces parallel to the $\xi_3 = C$ surface. For brevity, only the case with boundary surfaces parallel to the $\xi_3 = C$ surface will be derived here. It is assured that the Green's dyadic for the case of no boundary surface parallel to $\xi_3 = C$ surfaces may also be derived with the same technique, except for some minor modifications.

From the previous section it is seen that the eigenfunction solutions of the homogeneous Helmholtz equations may be written:

$$\left.\begin{aligned} \vec{M}_{mn} &= \nabla_\perp \psi_{mn} \times \vec{a}_3 \\ \vec{N}_{mn} &= \nabla_\perp \frac{\partial \chi_{mn}}{\partial \xi_3} - \nabla_\perp^2 \chi_{mn} \vec{a}_3 \\ \vec{L} &= \nabla_\perp \phi_{mn} + \frac{\partial \phi_{mn}}{\partial \xi_3} \vec{a}_3 \end{aligned}\right\} \qquad (3\text{-}4)$$

The index m in Eq. (3-4) designates the eigenvalue index arising from the separation of ξ_3, whereas the index n designates the eigenvalue arising from the separation of the $\perp$ coordinates; it depends upon the types of boundary in the $\perp$ direction. If it is an open boundary, n is a single index; on the other hand, if the boundary is a closed one, then n is a double index. In such cases it might be more appropriate to replace n by two indices l, n.

Since the scalars ψ_{mn}, χ_{mn}, and ϕ_{mn} are related, Eq. (3-4) can further be represented in the following form:

$$
\left.
\begin{aligned}
\vec{M} &\simeq \nabla_{\perp}\varphi_{mn}(\xi_1,\xi_2)\times\vec{a}_3 f_m(\xi_3) \\
\vec{N} &\simeq \nabla_{\perp}\varphi_{mn}(\xi_1,\xi_2)\frac{d}{d\xi_3}g_m(\xi_3) - \nabla_{\perp}^2\varphi_{mn}(\xi_1,\xi_2)g_m(\xi_3)\vec{a}_3 \\
\vec{L} &\simeq \nabla_{\perp}\varphi_{mn}(\xi_1,\xi_2)f_m(\xi_3) + \varphi_{mn}(\xi_1,\xi_2)\frac{d}{d\xi_3}f_m(\xi_3)\vec{a}_3
\end{aligned}
\right\} \tag{3-5}
$$

In view of the forms appearing in Eq. (3-5), it may be assumed that the Green's dyadic takes the form:

$$
\bar{\bar{G}} = \bar{\bar{G}}_M + \bar{\bar{G}}_N + \bar{\bar{G}}_L \tag{3-6}
$$

where

$$
\bar{\bar{G}}_M = \sum_{m,n}\left\{(\nabla_{\perp}\varphi_{mn}\times\vec{a}_3)f_m\right\}\vec{F}_{mn}(\xi_1^\circ,\xi_2^\circ,\xi_3^\circ) \tag{3-7 a}
$$

$$
\bar{\bar{G}}_N = \sum_{m,n}\left\{\nabla_{\perp}\varphi_{mn}\frac{dg_m}{d\xi_3} - \nabla_{\perp}^2\varphi_{mn}g_m\vec{a}_3\right\}\vec{G}_{mn}(\xi_1^\circ,\xi_2^\circ,\xi_3^\circ) \tag{3-7 b}
$$

$$
\bar{\bar{G}}_L = \sum_{m,n}\left\{\nabla_{\perp}\varphi_{mn}f_m + \varphi_{mn}\frac{df_m}{d\xi_3}\vec{a}_3\right\}\vec{H}_{mn}(\xi_1^\circ,\xi_2^\circ,\xi_3^\circ) \tag{3-7 c}
$$

where $\vec{F}_{mn}$, $\vec{G}_{mn}$, and $\vec{H}_{mn}$ are function of source coordinates only. The ξ_3 dependent functions, f_m and g_m, are the two independent solutions of Eq. (2-13); they must be related by either set of the equations:

$$
\left.
\begin{aligned}
\frac{dg_m}{d\xi_3} &= \pm k_m f_m \\
\frac{df_m}{d\xi_3} &= \mp k_m g_m
\end{aligned}
\right\} \quad \text{for closed boundary in } \xi_3 \tag{3-8 a}
$$

or

$$\left.\begin{aligned} \frac{dg_m}{d\xi_3} &= \pm j k_m f_m \\ \frac{df_m}{d\xi_3} &= \pm j k_m g_m \end{aligned}\right\} \text{ for open boundary in } \xi_3 \tag{3-8 b}$$

Since a closed boundary in ξ_3 is being considered, Eq. (3-8 a) will be adopted here, and for simplicity only the lower sign is used although the upper sign will yield the same result. It is also observed that although the vectors $\vec{M}_{mn}$, $\vec{N}_{mn}$, and $\vec{L}_{mn}$ are not necessarily orthogonal in space, but $\nabla_{\perp}\varphi_{mn}\times\vec{a}_3$, $\nabla_{\perp}\varphi_{mn}$, and $\vec{a}_3$ are three orthogonal vectors. If one defines a unit vector $\vec{b}$, and a two-variable-dependent function $\rho_{mn}(\xi_1,\xi_2)$, such that:

$$\nabla_{\perp}\varphi_{mn}(\xi_1,\xi_2) = \rho_{mn}(\xi_1,\xi_2)\,\vec{b} \tag{3-9}$$

then the unit vectors $\vec{b}$, $\vec{b}\times\vec{a}_3$, and $\vec{a}_3$ are mutually orthogonal in space. Multiplication of the unit vectors $\vec{b}$, $\vec{b}\times\vec{a}_3$, and $\vec{a}_3$ in turn into Eq. (3-3) yields a set of three mutually orthogonal equations:

$$\sum_{m,n}\left\{-(\nabla^2+k_{\perp})(\rho_{mn}f_m)\vec{F}_{mn} - k_T(\rho_{mn}f_m)(k_m\vec{G}_{mn}) + k_T(\rho_{mn}f_m)\vec{H}_{mn}\right\} = \vec{b}\times\vec{a}_3\,\delta(\vec{r}-\vec{r}_0) \tag{3-10}$$

$$\sum_{m,n}\left\{k_T(\rho_{mn}f_m)\vec{F}_{mn} - (\nabla^2+k_{\perp})(\rho_{mn}f_m)(k_m\vec{G}_{mn}) + k_{\perp}(\rho_{mn}f_m)\vec{H}_{mn}\right\} = \vec{b}\,\delta(\vec{r}-\vec{r}_0) \tag{3-11}$$

$$\sum_{m,n}\left\{(\nabla^2+k_{\parallel})(\nabla^2+k_m^2)(\varphi_{mn}g_m)(k_m\vec{G}_{mn}) - k_{\parallel}k_m^2(\varphi_{mn}g_m)\vec{H}_{mn}\right\} = k_m\vec{a}_3\,\delta(\vec{r}-\vec{r}_0) \tag{3-12}$$

In Equations (3-10) to (3-12), the relation given by Eq. (3-7) and Eq. (3-8) has been substituted. The operator ∇^2 is a three dimensional operator operating on the observer coordinate functions only, i.e., $(\rho_{mn} f_m)$ or $(\varphi_{mn} g_m)$. When later in the derivation operation on the source coordinate is needed, the operators will be distinguished by a superscript o, for example ∇^{o}.

In order to express the ξ_3^{o} dependent functions explicitly, and to express the source coordinate functions in component form $\vec{F}_{mn}$, $\vec{G}_{mn}$, and $\vec{H}_{mn}$ may be written.

$$\left.\begin{aligned}
\vec{F}_{mn} &= F^{\times}_{mn}(\xi_1^{o},\xi_2^{o})\, f^{\times}_m(\xi_3^{o})\, \vec{b}\times\vec{a}_3 + F^{\perp}_{mn}(\xi_1^{o},\xi_2^{o})\, f^{\perp}_m(\xi_3^{o})\, \vec{b} + F^{\parallel}_{mn}(\xi_1^{o},\xi_2^{o})\, f^{\parallel}_m(\xi_3^{o})\, \vec{a}_3 \\
\vec{G}_{mn} &= G^{\times}_{mn}(\xi_1^{o},\xi_2^{o})\, g^{\times}_m(\xi_3^{o})\, \vec{b}\times\vec{a}_3 + G^{\perp}_{mn}(\xi_1^{o},\xi_2^{o})\, g^{\perp}_m(\xi_3^{o})\, \vec{b} + G^{\parallel}_{mn}(\xi_1^{o},\xi_2^{o})\, g^{\parallel}_m(\xi_3^{o})\, \vec{a}_3 \\
\vec{H}_{mn} &= H^{\times}_{mn}(\xi_1^{o},\xi_2^{o})\, h^{\times}_m(\xi_3^{o})\, \vec{b}\times\vec{a}_3 + H^{\perp}_{mn}(\xi_1^{o},\xi_2^{o})\, h^{\perp}_m(\xi_3^{o})\, \vec{b} + H^{\parallel}_{mn}(\xi_1^{o},\xi_2^{o})\, h^{\parallel}_m(\xi_3^{o})\, \vec{a}_3
\end{aligned}\right\} \qquad (3\text{-}13)$$

After some vector manipulations, Eq. (3-10) to (3-12) are broken into a set of nine sixth-order equations:

$$
\begin{aligned}
&\sum_{m,n} \mathcal{M} (\rho_{mn} f_m)^2 (\varphi_{mn} g_n) F^{x}_{mn} f^{x}_m = \sum_{m,n} \left\{ k_{\parallel} k_m^2 \nabla^2 - k_{\perp} \nabla^2 (\nabla^2 + k_{\parallel} + k_m^2) \right\} (\rho_{mn} f_m)(\varphi_{mn} g_m) \delta(\vec{r} - \vec{r}_0) \\
&\sum_{m,n} \mathcal{M} (\rho_{mn} f_m)^2 (\varphi_{mn} g_m) F^{\perp}_{mn} f^{\perp}_m = \sum_{m,n} -k_T \nabla^2 (\nabla^2 + k_{\parallel} + k_m^2)(\rho_{mn} f_m)(\varphi_{mn} g_m) \delta(\vec{r} - \vec{r}_0) \\
&\sum_{m,n} \mathcal{M} (\rho_{mn} f_m)^2 (\varphi_{mn} g_m) F^{\parallel}_{mn} f^{\parallel}_m = \sum_{m,n} k_T k_m \nabla^2 (\rho_{mn} f_m)^2 \delta(\vec{r} - \vec{r}_0) \\
&\sum_{m,n} \mathcal{M} (\rho_{mn} f_m)(\varphi_{mn} g_m) G^{x}_{mn} g^{x}_m = \sum_{m,n} k_T k_{\parallel} k_m (\varphi_{mn} g_m) \delta(\vec{r} - \vec{r}_0) \\
&\sum_{m,n} \mathcal{M} (\rho_{mn} f_m)(\varphi_{mn} g_m) G^{\perp}_{mn} g^{\perp}_m = \sum_{m,n} -k_{\parallel} k_m (\nabla^2 + k_{\perp})(\varphi_{mn} g_m) \delta(\vec{r} - \vec{r}_0) \qquad (3\text{-}14) \\
&\sum_{m,n} \mathcal{M} (\rho_{mn} f_m)(\varphi_{mn} g_m) G^{\parallel}_{mn} g^{\parallel}_m = \sum_{m,n} \left\{ k_T^2 + k_{\perp} (\nabla^2 + k_{\perp}) \right\} (\rho_{mn} f_m) \delta(\vec{r} - \vec{r}_0) \\
&\sum_{m,n} \mathcal{M} (\rho_{mn} f_m)(\varphi_{mn} g_m) H^{x}_{mn} h^{x}_m = \sum_{m,n} k_T (\nabla^2 + k_{\parallel})(\nabla^2 + k_m^2)(\varphi_{mn} g_m) \delta(\vec{r} - \vec{r}_0) \\
&\sum_{m,n} \mathcal{M} (\rho_{mn} f_m)(\varphi_{mn} g_m) H^{\perp}_{mn} h^{\perp}_m = \sum_{m,n} -(\nabla^2 + k_{\parallel})(\nabla^2 + k_{\perp})(\nabla^2 + k_m^2)(\varphi_{mn} g_m) \delta(\vec{r} - \vec{r}_0) \\
&\sum_{m,n} \mathcal{M} (\rho_{mn} f_m)(\varphi_{mn} g_m) H^{\parallel}_{mn} h^{\parallel}_m = \sum_{m,n} k_m \left\{ k_T^2 + (\nabla^2 + k_{\perp})^2 \right\} (\rho_{mn} f_m) \delta(\vec{r} - \vec{r}_0)
\end{aligned}
$$

where the operator $\mathcal{M}$ is also an observer coordinate operator; it can be considered to be operating on any one of the two or three observer coordinate functions immediately to its right. Written in its entirety, $\mathcal{M}$ is

$$
\mathcal{M} = (\nabla^2 + k_{\parallel})(\nabla^2 + k_m^2) \left\{ k_T^2 + k_{\perp} (\nabla^2 + k_{\perp}) \right\} - k_{\parallel} k_m^2 \left\{ k_T^2 + (\nabla^2 + k_{\perp})^2 \right\} \qquad (3\text{-}15)
$$

Comparison of Eqs. (3-15) and (2-14), $\mathcal{M}_l$ may further be simplified into the form:

$$\mathcal{M}_l = \begin{cases} k_\perp T^2(T^2 - T_2^2)(\nabla^2 + T^2) \quad , & \text{if} \quad T^2 \to T_1^2 \\ \\ k_\perp T^2(T^2 - T_1^2)(\nabla^2 + T^2) \quad , & \text{if} \quad T^2 \to T_2^2 \end{cases} \tag{3-16}$$

where T_1^2 and T_2^2 are the two non-trivial roots of Eq. (2-14), and the relation of Eq. (2-11) has been used in arriving at Eq. (3-16). By the same token, all operators ∇^2 to the right of the equality sign in Eq. (3-14) are replaced by $(-T^2)$. Substitute Eq. (3-16) into Eq. (3-14) and drop out the functions common to both sides of the equality sign. Multiply both sides by f_m^* or g_m^*, whichever one is appropriate. Then integrate over the entire bounded ξ_3 space, utilizing the orthogonal properties of the eigenfunctions f_m and g_m :

$$\int f_m f_m^* \, dv_{\xi_3} = \Lambda_{f_m}^2$$

$$\int g_m g_m^* \, dv_{\xi_3} = \Lambda_{g_m}^2$$

where $\Lambda_{f_m}^2$ and $\Lambda_{g_m}^2$ are normalizing factors. The asterisk indicates the complex conjugate is employed. The integration yields distinct solutions for the ξ_3^0 dependent functions.

$$\left.\begin{aligned} \mathcal{F}_m(\xi_3^0) &= \mathcal{G}_m(\xi_3^0) = \frac{1}{\Lambda_{f_m}^2} f_m^*(\xi_3^0) \\ \\ \mathcal{G}_m(\xi_3^0) &= \frac{1}{\Lambda_{g_m}^2} g_m^*(\xi_3^0) \end{aligned}\right\} \tag{3-17}$$

The remains of Eq. (3-14) are two dimensional equations:

$$\left(\nabla_\perp^2 + T^2 - k_m^2\right)\rho_{mn} F_{mn}^{\times} = \frac{-1}{\eta}\left\{k_\perp\left(T^2 - k_\parallel - k_m^2\right) + k_\parallel k_m^2 T^2\right\}\delta(\vec{r}_\perp - \vec{r}_{0\perp}) \qquad \text{(3-18 a)}$$

$$\left(\nabla_\perp^2 + T^2 - k_m^2\right)\rho_{mn} F_{mn}^{\perp} = \frac{-1}{\eta} k_T T^2\left(T^2 - k_\parallel - k_m^2\right)\delta(\vec{r}_\perp - \vec{r}_{0\perp}) \qquad \text{(3-18 b)}$$

$$\left(\nabla_\perp^2 + T^2 - k_m^2\right)\varphi_{mn} F_{mn}^{\parallel} = -\frac{k_T k_m T^2}{\eta}\delta(\vec{r}_\perp - \vec{r}_{0\perp}) \qquad \text{(3-18 c)}$$

$$\left(\nabla_\perp^2 + T^2 - k_m^2\right)\rho_{mn} G_{mn}^{\times} = \frac{k_T k_\parallel k_m}{\eta}\delta(\vec{r}_\perp - \vec{r}_{0\perp}) \qquad \text{(3-18 d)}$$

$$\left(\nabla_\perp + T^2 - k_m^2\right)\rho_{mn} G_{mn}^{\perp} = \frac{k_\parallel k_m}{\eta}\left(T^2 - k_\perp\right)\delta(\vec{r}_\perp - \vec{r}_{0\perp}) \qquad \text{(3-18 e)}$$

$$\left(\nabla_\perp^2 + T^2 - k_m^2\right)\varphi_{mn} G_{mn}^{\parallel} = \frac{1}{\eta}\left\{k_T^2 - k_\perp\left(T^2 - k_\perp\right)\right\}\delta(\vec{r}_\perp - \vec{r}_{0\perp}) \qquad \text{(3-18 f)}$$

$$\left(\nabla_\perp^2 + T^2 - k_m^2\right)\rho_{mn} H_{mn}^{\times} = \frac{k_T}{\eta}\left(T^2 - k_\parallel\right)\left(T^2 - k_m^2\right)\delta(\vec{r}_\perp - \vec{r}_{0\perp}) \qquad \text{(3-18 g)}$$

$$\left(\nabla_\perp^2 + T^2 - k_m^2\right)\rho_{mn} H_{mn}^{\perp} = \frac{1}{\eta}\left(T^2 - k_\parallel\right)\left(T^2 - k_\perp\right)\left(T^2 - k_m^2\right)\delta(\vec{r}_\perp - \vec{r}_{0\perp}) \qquad \text{(3-18 h)}$$

$$\left(\nabla_\perp^2 + T^2 - k_m^2\right)\varphi_{mn} H_{mn}^{\parallel} = \frac{k_m}{\eta}\left\{k_T^2 + \left(T^2 - k_\perp\right)^2\right\}\delta(\vec{r}_\perp - \vec{r}_{0\perp}) \qquad \text{(3-18 i)}$$

where

$$\eta = \begin{cases} k_\perp T^2\left(T^2 - T_2^2\right) & \text{if} \quad T^2 \rightarrow T_1^2 \\ k_\perp T^2\left(T^2 - T_1^2\right) & \text{if} \quad T^2 \rightarrow T_2^2 \end{cases} \qquad \text{(3-19)}$$

Equations (3-18 c), (3-18 f) and (3-18 i) directly involve the function

$\varphi_{mn}(\xi_1, \xi_2)$; they can be represented by a symbolic equation:

$$\left(\nabla_\perp^2 + T^2 - k_m^2\right) g_{mn}(\xi_1, \xi_2 / \xi_1^0, \xi_2^0) = (Const.)\, \delta(\xi_1 - \xi_1^0)\delta(\xi_2 - \xi_2^0) \qquad (3\text{-}20)$$

One has reduced the problem to that of searching for an appropriate two-dimensional scalar Green's function. The problem is thus greatly simplified.

The remaining equations in (3-18) all involve $\rho_{mn}(\xi_1, \xi_2)$. Since the function ρ_{mn} comes from $\nabla_\perp \varphi_{mn}$, one may expect that the solutions for Eq. (3-20) may somehow be related to the solutions of the remaining six equations in Eq. (3-18). Indeed, it is so when it is observed that if φ_{mn} satisfies Eq. (2-12) then $\nabla_\perp \varphi_{mn}$ and $\nabla_\perp \varphi_{mn} \times \vec{a}_3$ must satisfy respectively:

$$\left(\nabla_\perp^2 + T^2 - k_m^2\right)\left(\nabla_\perp \varphi_{mn}\right) = 0$$

and

$$\nabla_\perp \times \nabla_\perp \times \left(\nabla_\perp \varphi_{mn} \times \vec{a}_3\right) - \left(T^2 - k_m^2\right)\left(\nabla_\perp \varphi_{mn} \times \vec{a}_3\right) = 0$$

If Eqs. (3-18 a), (3-18 d) and (3-18 g) are multiplied by the dyadic $(\vec{b} \times \vec{a}_3)(\vec{b} \times \vec{a}_3)$, the left-hand sides of these equations may be shown to be symbolically:

$$\begin{aligned}
&\left(\nabla_\perp^2 + T^2 - k_m^2\right) \rho_{mn} \eta_{mn}^{x} (\vec{b} \times \vec{a}_3)(\vec{b} \times \vec{a}_3) \\
&\quad = -\left(\nabla_\perp \times \nabla_\perp \times - T^2 - k_m^2\right)\left(\nabla_\perp \times \vec{a}_3 \varphi_{mn}\right)\left(\nabla_\perp^0 \times \vec{a}_3\, \eta_{mn}^{x}\right) \qquad (3\text{-}21) \\
&\quad = -\left(T^2 - k_m^2\right)\left(\nabla_\perp^2 + T^2 - k_m^2\right) \left| \varphi_{mn} \eta_{mn}^{x} \right| (\vec{b} \times \vec{a}_3)(\vec{b} \times \vec{a}_3)
\end{aligned}$$

Similarly, multiplying $\vec{b}\vec{b}$ to Eqs. (3-18 b), (3-18 e) and (3-18 h), their left-hand sides may be shown to be

$$\begin{aligned}(\nabla_\perp^2+\tau^2-k_m^2)\,\rho_{mn}\,\eta_{mn}^\perp\,\vec{b}\vec{b} &= (\nabla_\perp^2+\tau^2-k_m^2)\,\nabla_\perp\varphi_{mn}\,\nabla_\perp^\circ\,\eta_{mn}^\times \\ &= (\tau^2-k_m^2)(\nabla_\perp^2+\tau^2-k_m^2)\left|\varphi_{mn}\,\eta_{mn}\right|\vec{b}\vec{b}\end{aligned} \qquad (3\text{-}22)$$

The two bars bracketing a function indicate only the scalar is being considered.

Now, if $\varphi_{mn}\eta_{mn}^\times$ and $\varphi_{mn}\eta_{mn}^\perp$ are identified with $g_{mn}(\xi_1\xi_2/\xi_1^\circ\xi_2^\circ)$, then all nine equations in Eq. (3-18) have been reduced to one symbolic equation (3-20). Exact solutions of Eq. (3-20) depend upon the coordinate systems employed and the type of boundary considered. In general, it can be written symbolically:

$$g_{mn}=\begin{cases}\dfrac{(\text{Const.})}{\Lambda_n^2\Lambda_\ell^2}\,\varphi_{mn\ell}(\xi_1,\xi_2)\,\tilde{\varphi}_{mn\ell}(\xi_1^\circ,\xi_2^\circ) & \text{for closed boundary in } \perp \\[2ex] \dfrac{(\text{Const.})}{\nu^2}\,\varphi_{mn}(\xi_1,\xi_2)\,\tilde{\varphi}_{mn}(\xi_1^\circ,\xi_2^\circ) & \text{for open boundary in } \perp\end{cases} \qquad (3\text{-}20)$$

In the case of a closed boundary, $\tilde{\varphi}_{mn\ell}=\varphi_{mn\ell}^*$ is the complex conjugate of $\varphi_{mn\ell}$, and Λ_n^2 and Λ_ℓ^2 are the two normalization factors. In the case of an open boundary, φ_{mn} and $\tilde{\varphi}_{mn}$ are the two independent solutions of Eq. (2-12) and ν^2 is a constant involving the Wronskian of the two independent solutions. With Eq. (3-20) solved, the Green's dyadic may then be readily obtained:

$$
\begin{aligned}
\overline{\overline{G}}_M = \sum_{m,n} & \left\{ \nabla_\perp \varphi_{mn}(\xi_1,\xi_2) \times \vec{a}_3 f_m(\xi_3) \right\} \left\{ \frac{1}{\eta v^2 \lambda_{fm}^2} \left(\frac{k_\perp (T^2 - k_\parallel - k_m^2) + k_\parallel k_m^2 T^2}{T^2 - k_m^2} \right) \times \right. \\
& \nabla_\perp^\circ \tilde{\varphi}_{mn}(\xi_1^\circ,\xi_2^\circ) \times \vec{a}_3 f_m^*(\xi_3^\circ) - \frac{k_T T^2}{\eta v^2 \lambda_{fm}^2} \left(\frac{T^2 - k_\parallel - k_m^2}{T^2 - k_m^2} \right) \nabla_\perp^\circ \tilde{\varphi}_{mn}(\xi_1^\circ,\xi_2^\circ) f_m^*(\xi_3^\circ) \\
& \left. - \frac{k_T k_m T^2}{\eta v^2 \lambda_{gm}^2} \tilde{\varphi}_{mn}(\xi_1^\circ,\xi_2^\circ) g_m^*(\xi_3^\circ) \vec{a}_3 \right\} \\
\overline{\overline{G}}_N = \sum_{m,n} & \left\{ \nabla_\perp \varphi_{mn}(\xi_1,\xi_2) \frac{d g_m(\xi_3)}{d \xi_3} + (T^2 - k_m^2) \varphi_{mn}(\xi_1,\xi_2) g_m(\xi_3) \vec{a}_3 \right\} \times \\
& \left\{ - \frac{1}{\eta v^2 \lambda_{fm}^2} \frac{k_T k_\parallel k_m}{T^2 - k_m^2} \nabla_\perp^\circ \tilde{\varphi}_{mn}(\xi_1^\circ,\xi_2^\circ) \times \vec{a}_3 f_m^*(\xi_3^\circ) \right. \\
& \left. + \frac{k_\parallel k_m}{\eta v^2 \lambda_{fm}^2} \left(\frac{T^2 - k_\perp}{T^2 - k_m^2} \right) \nabla_\perp^\circ \tilde{\varphi}_{mn}(\xi_1^\circ,\xi_2^\circ) f_m^*(\xi_3^\circ) + \frac{k_T^2 - k_\perp (T^2 - k_\perp)}{\eta v^2 \lambda_{gm}^2} \tilde{\varphi}_{mn}(\xi_1^\circ,\xi_2^\circ) g_m^*(\xi_3^\circ) \vec{a}_3 \right\} \qquad (3\text{-}25) \\
\overline{\overline{G}}_L = \sum_{m,n} & \left\{ \nabla_\perp \varphi_{mn}(\xi_1,\xi_2) f_m(\xi_3) + \varphi_{mn}(\xi_1,\xi_2) \frac{d f_m(\xi_3)}{d \xi_3} \right\} \times \\
& \left\{ - \frac{k_T (T^2 - k_\parallel)}{\eta v^2 \lambda_{fm}^2} \nabla_\perp^\circ \tilde{\varphi}_{mn}(\xi_1^\circ,\xi_2^\circ) \times \vec{a}_3 f_m^*(\xi_3^\circ) + \frac{(T^2 - k_\parallel)(T^2 - k_\perp)}{\eta v^2 \lambda_{fm}^2} \nabla_\perp^\circ \tilde{\varphi}_{mn}(\xi_1^\circ,\xi_2^\circ) f_m^*(\xi_3^\circ) \right. \\
& \left. + \frac{k_m}{\eta v^2 \lambda_{gm}^2} \left[k_T^2 + (T^2 - k_\perp)^2 \right] \tilde{\varphi}_{mn}(\xi_1^\circ,\xi_2^\circ) g_m^*(\xi_3^\circ) \vec{a}_3 \right\}
\end{aligned}
$$

In circular cylindrical coordinates with open boundary in r and θ , the solution for Eq. (3-20) is:

$$
g_{mn}(\vec{r}\,|\,\vec{r}_0) = (\text{Const.})\, e^{in(\theta - \theta_0)} \begin{cases} J_n(\kappa r) H_n^{(2)}(\kappa r_0) & ; \quad r \le r_0 \\ J_n(\kappa r_0) H_n^{(2)}(\kappa r) & ; \quad r \ge r_0 \end{cases} \qquad (3\text{-}26)
$$

$$\kappa^2 = T^2 - k_m^2$$

Identifying $f_m(\xi_3)$ with $\sin k_m z$ and $g_m(\xi_3)$ with $\cos k_m z$ in line with

Eq. (2-17), the Green's dyadic is: (for $r \leq r_0$ case only)

$$
\begin{aligned}
\vec{\vec{G}} = & \sum_{m,n} \frac{k_\perp(T^2-k_{\parallel}-k_m^2)+k_{\parallel}k_m^2T^2}{\pi\eta\,(T^2-k_m^2)} \left\{\nabla_\perp\left[J_n(\kappa r)e^{in\theta}\right]\times\vec{a}_3\right\}\left\{\nabla_\perp^0\left[H_n^{(2)}(\kappa r_0)e^{-in\theta_0}\right]\times\vec{a}_3\right\}\sin k_m z\,\sin k_m z_0 \\
& + \sum_{m,n} \frac{-(T^2-k_\perp)}{\pi\eta}\left[(T^2-k_{\parallel})-\frac{k_{\parallel}k_m^2}{T^2-k_m^2}\right]\left\{\nabla_\perp\left[J_n(\kappa r)e^{in\theta}\right]\right\}\left\{\nabla_\perp^0\left[H_n^{(2)}(\kappa r_0)e^{-in\theta_0}\right]\right\}\sin k_m z\,\sin k_m z_0 \\
& + \sum_{m,n} \frac{T^2}{\pi\eta}\left[k_T^2-(k_\perp-k_m^2)(T^2-k_\perp)\right]\left\{J_n(\kappa r)H_n^{(2)}(\kappa r_0)e^{in(\theta-\theta_0)}\cos k_m z\,\cos k_m z_0\,\vec{a}\vec{a}\right\} \\
& - \sum_{m,n} \frac{k_T T^2}{\pi\eta}\left(\frac{T^2-k_{\parallel}-k_m^2}{T^2-k_m^2}\right)\left\{\left[\nabla_\perp J_n(\kappa r)e^{in\theta}\times\vec{a}_3\right]\left[\nabla_\perp^0 H_n^{(2)}(\kappa r_0)e^{-in\theta_0}\right]\sin k_m z\,\sin k_m z_0\right. \\
& \qquad \left. + \left[\nabla_\perp J_n(\kappa r)e^{in\theta}\right]\left[\nabla_\perp^0 H_n^{(2)}(\kappa r_0)e^{-in\theta_0}\times\vec{a}_3\right]\sin k_m z\,\sin k_m z_0\right\} \qquad (3\text{-}27) \\
& - \sum_{m,n} \frac{k_T k_m T^2}{\pi\eta}\left\{\left[\nabla_\perp J_n(\kappa r)e^{in\theta}\times\vec{a}_3\right]\left[H_n^{(2)}(\kappa r_0)e^{-in\theta_0}\vec{a}_3\right]\sin k_m z\,\cos k_m z_0\right. \\
& \qquad \left. + \left[J_n(\kappa r)e^{in\theta}\vec{a}_3\right]\left[\nabla_\perp^0 H_n^{(2)}(\kappa r_0)e^{-in\theta_0}\times\vec{a}_3\right]\cos k_m z\,\sin k_m z_0\right\} \\
& + \sum_{m,n} \frac{k_m T^2(T^2-k_\perp)}{\pi\eta}\left\{\left[\nabla_\perp J_n(\kappa r)e^{in\theta}\right]\left[H_n^{(2)}(\kappa r_0)e^{-in\theta_0}\vec{a}_3\right]\sin k_m z\,\cos k_m z_0\right. \\
& \qquad \left. + \left[J_n(\kappa r)e^{in\theta}\vec{a}_3\right]\left[\nabla_\perp^0 H_n^{(2)}(\kappa r_0)e^{-in\theta_0}\right]\cos k_m z\,\sin k_m z_0\right.
\end{aligned}
$$

III. Remarks and Proposed Future Studies

The results obtained in the preceding sections enable one to extend considerably his scope of investigations in the problems of waves propagation in plasma. For example, waves generated by a current sheet source of finite width may be analyzed in elliptical cylindrical coordinates. Waves generated by a semi-infinite sheet of current will require analysis in parabolic cylindrical coordinates. Investigation of problems similar to these were previously avoided due to inadequate tools.

The work to be carried out in the future will be concentrated on the investigation of the properties of radially propagated waves; especially the properties of the waves near the source region.

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